Lecture 2

§1.1–1.2 (cont.)

Recall: Rough speaking, a D.E of order n

is $F(x, y, \cdots, \frac{dy^n}{dx^n}) = 0$ $\quad (*)$

Two definitions — explicit solution / implicit solution.

Def^n: By an explicit solution to $(*)$, we mean a function $y = \phi(x)$ that satisfies $(*)$.

E.g

① Given D.E

$$\frac{d^2y}{dx^2} + y = 0 \qquad (1)$$

Verify for $\forall A, B \in \mathbb{R}$,

$y = A\cos x + B\sin x$ is an explicit solution to (1).

② Find values of A, B such that

$y = A\cos x + B\sin x$ solves I.V.P:

$$\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 & \text{D.E} \\ y(0) = 1, \quad y'(0) = 2 & \text{initial condition} \end{cases}$$

A: ① (plug in $y = \phi(x)$ and verify the D.E holds)

Since $y = A\cos x + B\sin x$

$$\Rightarrow \frac{dy}{dx} = y' = -A\sin x + B\cos x$$

$$\frac{d^2y}{dx^2} = (y')' = -A\cos x - B\sin x$$

Hence
$$\frac{d^2y}{dx^2} + y = (\cancel{-A\cos x} - \cancel{B\sin x}) + (\cancel{A\cos x} + \cancel{B\sin x})$$
$$= 0$$

This verifies the D.E holds. and thus $y = A\cos x + B\sin x$ is an explicit soln.

② (Initial condition: $y(0)=1$, $y'(0)=2$)

Let $x=0$,

$$y(0) = A\cos 0 + B\sin 0$$
$$= A$$

cos 0 = 1
sin 0 = 0

$y(0)=1 \Rightarrow A = 1$

Recall $y' = -A\sin x + B\cos x$

Let $x=0 \Rightarrow$

$$y'(0) = B$$

$y'(0)=2 \Rightarrow B=2$

Hence $y = \cos x + 2\sin x$

explicit soln: $\underline{y=\phi(x)}$

Def^n: A relation/equation $G(x,y)=0$ is called an implicit solution if it gives one or more solutions to the D.E

E.g: verify $x^2+y^2=1$ gives an implicit solution for the D.E

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Remark: $x^2+y^2=1 \Rightarrow y^2=1-x^2 \Rightarrow$

$$y = \pm\sqrt{1-x^2}$$

(1) $y=\sqrt{1-x^2}$ (2) $y=-\sqrt{1-x^2}$

Note: In general, it might be hard to solve y out of the eqn $G(x,y)=0$

We, however, have a more general way to verify the implicit solution by using implicit differentiation

$y = y(x)$

Step 1: Regard y as a function of x and differentiate the eqn with respect to x.

differentiate $x^2 + y^2 = 1$ w.r.t to x

$\Rightarrow$ $$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{d}{dx}(y(x))^2 = 2y \cdot y'}$$

Step 2: Solve $\frac{dy}{dx}$ from what you get in step 1 if necessary:

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Overview: Chapter 2 discusses 3 ways to solve 1st order D.E.

Sec 2.2 separable D.Es.

A D.E is called separable if it can be rewritten in the form:

$$\frac{dy}{dx} = f(x)\cdot g(y).$$

Here f, g are allowed to be constant

(that is, RHS of the eqn is the product of two parts, where one part depends only on x, the other part depends only on y)

E.g Are the following D.E separable.

① $$\frac{dy}{dx} - y^2 = 0$$ ✓

$$\Rightarrow \frac{dy}{dx} = y^2 = \underbrace{1}_{f(x)} \cdot \underbrace{y^2}_{g(y)}$$

② $\frac{dy}{dx} - y^2 = x$ Not separable

$$\frac{dy}{dx} = x + y^2$$

$\overset{?}{=} f(x) \cdot g(y)$ Not possible

③ $\frac{dy}{dx} - xy - x - y = 1$ ✓ separable

$$\Rightarrow \quad \frac{dy}{dx} = xy + x + y + 1$$

$$= (1+x) + y(1+x)$$

$$= \underbrace{(1+x)}_{f(x)}\underbrace{(1+y)}_{g(y)}$$

Ideas to solve separable D.E.

$$\frac{dy}{dx} = f(x)\,g(y). \qquad (2)$$

Step 1: Check whether $g(y)=0$ gives a soln. to (2) (If $g(y)$ cannot be 0, then you don't need to do step 1)

Step 2: Suppose $g(y) \neq 0$.

$$(2) \Rightarrow \frac{dy}{g(y)} = f(x)\,dx$$

Then integrate $\Rightarrow$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

Calculus $\Rightarrow$ $G(y) = F(x) + C$

Remark 1. The above is an implicit solution (Not in the form $y = \phi(x)$)

2. Technically $G(y) + C_1 = F(x) + C_2$

$$\Rightarrow \quad G(y) = F(x) + \underbrace{(C_2 - C_1)}_{C}$$

E.g 1. solve $\dfrac{dy}{dx} = \dfrac{y-1}{x+3}$ (3)

Idea: $\dfrac{dy}{dx} = \dfrac{y-1}{x+3}$

$$= \underbrace{\frac{1}{x+3}}_{f(x)} \underbrace{(y-1)}_{g(y)}$$

Step 1: Check whether $y-1=0$ gives a soln.

$y-1=0 \Rightarrow y=1$ Then $\dfrac{dy}{dx} = 0$;

LHS of (3) = 0,

RHS of (3) $= \dfrac{y-1}{x+3} = 0 \Rightarrow$ (3) holds

Hence $y=1$ is a soln to (3)

Step 2: Assume $y-1 \neq 0$. Move terms about y to LHS. move terms about x to RHS

$$\frac{dy}{y-1} = \frac{1}{x+3}\, dx$$

Integrate $\Rightarrow$ $\displaystyle\int \frac{dy}{y-1} = \int \frac{dx}{x+3}$

$\Rightarrow$

$\ln|y-1| = \ln|x+3| + C$

(Implicit soln)

Hint: a, b constant, $a \neq 0$

$$\int \frac{1}{at+b}\,dt = \frac{1}{a}\ln|at+b| + C$$

In particular, $a=1$

$$\int \frac{1}{t+b}\,dt = \ln|t+b| + C$$

verify by u-sub
"$u = at+b$"

To summarize, we have the following solns:

(1) $y = 1$

(2) $\ln|y-1| = \ln|x+3| + C$

E.g2: solve the following I.V.P

$$\begin{cases} \frac{dy}{dx} = \frac{y-1}{x+3} & (3) \\ y(-1) = 0 & (4) \end{cases}$$ Initial condition

Write the soln as an explicit soln $y=\phi(x)$.

A: Recall by the above E.g 1, (3) has two kinds of solns:

① $y=1$ $\Leftrightarrow$ $y=1$ everywhere

② $\ln|y-1| = \ln|x+3| + C$.

For ①, it can never satisfy $y(-1)=0$

For ②, since $y(-1)=0$, that is,

when $x=-1$, $y=0$, we plug in

$(x,y)=(-1,0)$ to ② $\Rightarrow$

$$\ln|-1| = \ln|-1+3| + C$$

$$\Rightarrow \underbrace{\ln 1}_{0} = \ln 2 + C$$

$$\Rightarrow C = -\ln 2$$

Hence the soln is

$$\ln|y-1| = \ln|x+3| - \ln 2$$

We raise both side to exp:

$$\Rightarrow e^{\ln|y-1|} = e^{\ln|x+3| - \ln 2}$$

$$e^{\ln a} = a \quad (a>0)$$

$$e^{A-B} = \frac{e^A}{e^B}$$

$$\Rightarrow |y-1| = \frac{e^{\ln|x+3|}}{e^{\ln 2}}$$

$$= \frac{1}{2}|x+3|$$

$$\Rightarrow \underbrace{|y-1|}_{A} = \underbrace{|\tfrac{1}{2}(x+3)|}_{B}$$

$$\boxed{\begin{aligned}&|A| = |B|\\ \Rightarrow\ &A = \pm B\end{aligned}}$$

$\Rightarrow$

$$y-1 = \pm\frac{1}{2}(x+3)$$

$\Rightarrow$

(1) $y-1 = +\frac{1}{2}(x+3)$

(2) $y-1 = -\frac{1}{2}(x+3)$

But $y(-1) = 0$ (i.e when $x=-1$, $y=0$)

$\Rightarrow$ (1) is NOT possible

(At $(x,y) = (-1,0)$, $\begin{cases} \text{LHS of (1)} = -1 \\ \text{RHS of (1)} = \frac{1}{2}(-1+3) = 1 \end{cases}$)

And LHS of (2) $= -1$

RHS of (2) $= -\frac{1}{2}(-1+3) = -1$

$\Rightarrow$ (2) satisfies the initial condition.

Hence only (2) is the soln:

$$y - 1 = -\frac{1}{2}(x+3)$$

or $y = -\frac{1}{2}x - \frac{1}{2}$